CHAPTER V.

NORMAL VARIABILITY.

Schemes of Deviations.—Normal Curve of Distribution.—Comparison of the observed with the Normal Curve.—The value of a single Deviation at a known Grade determines a Normal Scheme of Deviations.—Two Measures at two known Grades determine a Normal Scheme of Measures.—The Charms of Statistics.—Mechanical illustration of the Cause of the Curve of Frequency.—Order in apparent Chaos.—Problems in the Law of Error.

Schemes of Deviations.—We have now seen how easy it is to represent the distribution of any quality among a multitude of men, either by a simple diagram or by a line containing a few figures. In this chapter it will be shown that a considerably briefer description is approximately sufficient.

Every measure in a Scheme is equal to its Middlemost, or Median value, or M, *plus* or *minus* a certain Deviation from M. The Deviation, or "Error" as it is technically called, is *plus* for all grades above 50°, zero for 50°, and *minus* for all grades below 50°. Thus if (±D) be the deviation from M in any particular case, every measure in a Scheme may be expressed in the form of M + (±D). If M = O, or if it is subtracted from every measure, the residues which are the different values of (±D) will form a Scheme by themselves. Schemes may therefore be made of Deviations as well as of Measures, and one of the former is seen in the upper part of Fig. 6, page 40. It is merely the upper portion of the corresponding Scheme of Measures, in which the axis of the curve plays the part of the base.

A strong family likeness runs between the 18 different Schemes of Deviations that may be respectively derived from the data in the 18 lines of Table 2. If the slope of the curve in one Scheme is steeper than that of another, we need only to fore-shorten the steeper Scheme, by inclining it away from the line of sight, in order to reduce its apparent steepness and to make it look almost identical with the other. Or, better still, we may select appropriate vertical scales that will enable all the Schemes to be drawn afresh with a uniform slope, and be made strictly comparable.

Suppose *that we have only two Schemes*, A. *and* B., that we wish to compare. Let $L._1$, $L._2$ be the lengths of the perpendiculars at two specified grades in Scheme A., and $K._1$ $K._2$ the lengths of those at the same grades in Scheme B.; then if every one of the data from which Scheme B. was drawn be multiplied by $\frac{L._1 - L._2}{K._1 - K._2}$, a series of transmuted data will be obtained for drawing a new Scheme B′., on such a vertical scale that its *general slope between the selected grades shall be the same as in Scheme* A. For practical convenience the selected Grades will be always those of 25° and 75°. They stand at the first and third quarterly divisions of the base, and are therefore easily found by a pair of compasses. They are also well placed to afford a fair criterion of the general slope of the Curve. If we call the perpendicular at 25°, $Q._1$; and that at 75°, $Q._2$, then the unit by which every Scheme will be defined is its value of $\frac{1}{2}(Q._2 - Q._1)$, and will be called its Q. As the M measures the Average Height of the curved boundary of a Scheme, so the Q measures its general slope. When we wish to transform many different Schemes, numbered I., II., III., &c., whose respective values of Q are q_1, q_2, q_3, &c., to others whose values of Q are in each case equal to q_0, then all the data from which Scheme I. was drawn, must be multiplied by $\frac{q_0}{q_1}$; those from which Scheme II. was drawn, by $\frac{q_0}{q_2}$, and so on, and new Schemes have to be constructed from these transmuted values.

Our Q has the further merit of being practically the same as the value which mathematicians call the "Probable Error," of which we shall speak further on.

Want of space in Table 2 prevented the insertion of the measures at the Grades 25° and 75°, but those at 20° and 30° are given on the one hand, and those at 70° and 80° on the other, whose respective averages differ but little from the values at 25° and 75°. I therefore will use those four measures to obtain a value for our unit, which we will call Q′, to distinguish it from Q.

These are not identical in value, because the outline of the Scheme is a curved and not a straight line, but the difference between them is small, and is approximately the same in all Schemes. It will shortly be seen that $Q' = 1{\cdot}015 \times Q$ approximately; therefore a series of Deviations measured in terms of the large unit Q' are numerically smaller than if they had been measured in terms of the small unit (for the same reason that the numerals in 2, 3, &c., *feet* are smaller than those in the corresponding values of 24, 36, &c., *inches*), and they must be multiplied by 1.015 when it is desired to change them into a series having the smaller value of Q for their unit.

All the 18 Schemes of Deviation that can be derived from Table 2 have been treated on these principles, and the results are given in Table 3. Their general accordance with one another, and still more with the mean of all of them, is obvious.

Normal Curve of Distribution.—The values in the bottom line of Table 3, which is headed "Normal Values when $Q = 1$," and which correspond with minute precision to those in the line immediately above them, are not derived from observations at all, but from the well-known Tables of the "Probability Integral" in a way that mathematicians will easily understand by comparing the Tables 4 to 8 inclusive. I need hardly remind the reader that the Law of Error upon which these Normal Values are based, was excogitated for the use of astronomers and others who are concerned with extreme accuracy of measurement, and without the slightest idea until the time of Quetelet that they might be applicable to human measures. But Errors, Differences, Deviations, Divergencies, Dispersions, and individual Variations, all spring from the same kind of causes. Objects that bear the same name, or can be described by the same phrase, are thereby acknowledged to have common points of resemblance, and to rank as members of the same species, class, or whatever else we may please to call the group. On the other hand, every object has Differences peculiar to itself, by which it is distinguished from others.

This general statement is applicable to thousands of instances. The Law of Error finds a footing wherever the individual peculiarities are wholly due to the combined influence of a multitude of "accidents," in the sense in which that word has already been defined. All persons conversant with statistics are aware that this supposition brings Variability within the grasp of the laws of Chance, with the result that the relative frequency of Deviations of different amounts admits of being calculated, when those amounts are measured in terms of any self-contained unit of variability, such as our Q. The Tables 4 to 8 give the results of these purely mathematical calculations, and the Curves based upon them may with propriety be distinguished as "Normal." Tables 7 and 8 are based upon the familiar Table of the Probability Integral, given in Table 5, *vid* that in Table 6, in which the unit of variability is taken to be the "Probable Error" or our Q, and not the "Modulus." Then I turn Table 6 inside out, as it were, deriving the "arguments" for Tables 7 and 8 *from the entries in the body of Table 6*, and making other easily intelligible alterations.

Comparison of the Observed with the Normal Curve. —I confess to having been amazed at the extraordinary coincidence between the two bottom lines of Table 3, considering the great variety of faculties contained in the 18 Schemes; namely, three kinds of linear measurement, besides one of weight, one of capacity, two of strength, one of vision, and one of swiftness. It is obvious that weight cannot really vary at the same rate as height, even allowing for the fact that tall men are often lanky, but the theoretical impossibility is of the less practical importance, as the variations in weight are small compared to the weight itself. Thus we see from the value of Q in the first column of Table 3, that half of the persons deviated from their M by no more than 10 or 11 lbs., which is about one-twelfth part of the value of M. Although the several series in Table 3 run fairly well together, I should not have dared to hope that their irregularities would have balanced one another so beautifully as they have done. It has been objected to some of my former work, especially in *Hereditary Genius*, that I pushed the applications of the Law of Frequency of Error somewhat too far. I may have done so, rather by incautious phrases than in reality; but I am sure that, with the evidence now before us, the applicability of that law is more than justified within the reasonable limits asked for in the present book. I am satisfied to claim that the Normal Curve is a fair average representation of the Observed Curves during nine-tenths of their course; that is, for so much of them as lies between the grades of 5° and 95°. In particular, the agreement of the Curve of Stature with the Normal Curve is very fair, and forms a mainstay of my inquiry into the laws of Natural Inheritance.

It has already been said that mathematicians laboured at the law of Error for one set of purposes, and we are entering into the fruits of their labours for another. Hence there is no ground for surprise that their Nomenclature is often cumbrous and out of place, when applied to problems in heredity. This is especially the case with regard to their term of "Probable Error," by which they mean the value that one half of the Errors exceed and the other half fall short of. This is practically the same as our Q.[1] It is strictly the same whenever the two halves of the Scheme of Deviations to which it applies are symmetrically disposed about their common axis.

The term Probable Error, in its plain English interpretation of the *most* Probable Error, is quite misleading, for it is *not* that. The most Probable Error (as Dr. Venn has pointed out, in his *Logic of Chance*)

[1] The following little Table may be of service :—

Values of the different Constants when the Prob. Error is taken as unity, and their corresponding Grades.

Prob. Error	1·000 ;	corresponding Grades	25°·0,	75°·0
Modulus	2·097 ;	„ „	7°·9,	92°·1
Mean Error..............	1·183 ;	„ „	21°·2,	78°·8
Error of Mean Squares	1·483 ;	„ „	16°·0,	84°·0

is zero. This results from what was said a few pages back about the most probable measure in a Scheme being its M. In a Scheme of Errors the M is equal to 0, therefore the most Probable Error in such a Scheme is 0 also. It is astonishing that mathematicians, who are the most precise and perspicacious of men, have not long since revolted against this cumbrous, slip-shod, and misleading phrase. They really mean what I should call the Mid-Error, but their phrase is too firmly established for me to uproot it. I shall however always write the word Probable when used in this sense, in the form of "Prob."; thus "Prob. Error," as a continual protest against its illegitimate use, and as some slight safeguard against its misinterpretation. Moreover the term Probable Error is absurd when applied to the subjects now in hand, such as Stature, Eye-colour, Artistic Faculty, or Disease. I shall therefore usually speak of Prob. Deviation.

Though the value of our Q is the same as that of the Prob. Deviation, Q is not a convertible term with Prob. Deviation. We shall often have to speak of the one without immediate reference to the other, just as we speak of the diameter of the circle without reference to any of its properties, such as, if lines are drawn from its ends to any point in the circumference, they will meet at a right angle. The Q of a Scheme is as definite a phrase as the Diameter of a Circle, but we cannot replace Q in that phrase by the words Prob. Deviation, and speak of the Prob. Deviation of a Scheme, without doing some violence to language. We should have to express ourselves from another point of view, and at much greater length, and say "the Prob. Deviation of any, as yet unknown measure in the Scheme, from the Mean of all the measures from which the Scheme was constructed."

The primary idea of Q has no reference to the existence of a Mean value from which Deviations take place. It is half the difference between the measures found at the 25th and 75th Centesimal Grades. In this definition there is not the slightest allusion, direct or indirect, to the measure at the 50th Grade, which is the value of M. It is perfectly true that the measure at Grade 25° is M—Q, and that at Grade 75° is M + Q, but all this is superimposed upon the primary conception. Q stands essentially on its own basis, and has nothing to do with M. It will often happen that we shall have to deal with Prob: Deviations, but that is no reason why we should not use Q whenever it suits our purposes better, especially as statistical statements tend to be so cumbrous that every abbreviation is welcome.

The stage to which we have now arrived is this. It has been shown that the distribution of very different human qualities and faculties is approximately Normal, and it is inferred that with reasonable precautions we may treat them as if they were wholly so, in order to obtain approximate results. We shall thus deal with an entire Scheme of Deviations in terms of its Q, and with an entire Scheme of Measures in terms of its M and Q, just as we deal with an entire Circle in terms of its radius, or with an entire Ellipse in terms of its major and minor axes. We can also apply the various beautiful properties of the Law of Frequency of Error to the observed values of Q. In doing so, we act like woodsmen who roughly calculate the cubic contents of the trunk of a tree, by measuring its length, and its girth at either end, and submitting their measures to formulæ that have been deduced from the properties of ideally perfect straight lines and circles. Their results prove serviceable, although the trunk is only rudely straight and circular. I trust that my results will be yet closer approximations to the truth than those usually arrived at by the woodsmen.

The value of a single Deviation at a known Grade determines a Normal Scheme of Deviations.—When Normal Curves of Distribution are drawn within the same limits, they differ from each other only in their general slope; and the slope is determined if the value of the Deviation is given at any one specified Grade. It must be borne in mind that the width of the limits between which the Scheme is drawn, has no influence on the values of the Deviations at the various Grades, because the latter are proportionate parts of the base. As the limits vary in width, so do the intervals between the Grades. When measuring the Deviation at a specified Grade for the purpose of determining the whole Curve, it is of course convenient to adhere to the same Grade in all cases. It will be recollected that when dealing with the observed curves a few pages back, I used not one Grade but two Grades for the purpose, namely 25° and 75°; but in the Normal Curve, the *plus* and *minus* Deviations are equal in amount at all pairs of symmetrical distances on either side of grade 50°; therefore the Deviation at either of the Grades 25° or 75° is equal to Q, and suffices to define the entire Curve.

The reason why a certain value Q′ was stated a few pages back to be equal to 1·015 Q, is that the Normal Deviations at 20° and at 30°, (whose average we called Q′) are found in Table 8, to be 1·25 and 0·78; and similarly those at 70° and 60°. The average of 1·25 and 0·78 is 1·015, whereas the Deviation at 25° or at 75° is 1·000.

Two Measures at known Grades determine a Normal Scheme of Measures.—If we know the value of M as well as that of Q we know the entire Scheme. M expresses the mean value of all the objects contained in the group, and Q defines their variability. But if we know the Measures at any two specified Grades, we can deduce M and Q from them, and so determine the entire Scheme. The method of doing this is explained in the foot-note.[1]

[1] The following is a fuller description of the propositions in this and in the preceding paragraph:—

(1) In any Normal Scheme, and therefore approximately in an observed one, if the value of the Deviation is given at any *one* specified Grade the whole Curve is determined. Let D be the given Deviation, and d the tabular Deviation at the same Grade, as found in Table 8; then multiply every entry in Table 8 by $\frac{D}{d}$. As the tabular value of Q is 1, it will become changed into $\frac{D}{d}$.

The Charms of Statistics.—It is difficult to understand why statisticians commonly limit their inquiries to Averages, and do not revel in more comprehensive views. Their souls seem as dull to the charm of variety as that of the native of one of our flat English counties, whose retrospect of Switzerland was that, if its mountains could be thrown into its lakes, two nuisances would be got rid of at once. An Average is but a solitary fact, whereas if a single other fact be added to it, an entire Normal Scheme, which nearly corresponds to the observed one, starts potentially into existence.

Some people hate the very name of statistics, but I find them full of beauty and interest. Whenever they are not brutalised, but delicately handled by the higher methods, and are warily interpreted, their power of dealing with complicated phenomena is extraordinary. They are the only tools by which an opening can be cut

(2) If the Measures at any *two* specified Grades are given, the whole Scheme of Measures is thereby determined. Let A, B be the two given Measures of which A is the larger, and let a, b be the values of the tabular Deviations for the same Grades, as found in Table 8, not omitting their signs of *plus* or *minus* as the case may be.

Then the $\mathfrak{Q}$ of the Scheme $= \pm\frac{A-B}{a-b}$. (The sign of $\mathfrak{Q}$ is not to be regarded; it is merely a magnitude.)

$$\mathfrak{M} = A - a\,\mathfrak{Q}\text{; or } \mathfrak{M} = B - b\,\mathfrak{Q}.$$

Example: A, situated at Grade 55°, = 14·38
B, situated at Grade 5°, = 9·12

The corresponding tabular Deviations are :—$a = +0{\cdot}19$; $b = -2{\cdot}44$.

$$\text{Therefore } \mathfrak{Q} = \frac{14{\cdot}38 - 9{\cdot}12}{0{\cdot}19 + 2{\cdot}44} = \frac{5{\cdot}26}{2{\cdot}63} = 2{\cdot}0$$

$$\mathfrak{M} = 14.38 - 0{\cdot}19 \times 2 = 14{\cdot}0$$

$$\text{or } = 9{\cdot}12 + 2{\cdot}44 \times 2 = 14{\cdot}0$$

through the formidable thicket of difficulties that bars the path of those who pursue the Science of man.

Mechanical Illustration of the Cause of the Curve of Frequency.—The Curve of Frequency, and that of Distribution, are convertible : therefore if the genesis of either of them can be made clear, that of the other becomes also intelligible. I shall now illustrate the origin of the Curve of Frequency, by means of an apparatus shown in Fig. 7, that mimics in a very pretty way the conditions

FIG. 7.

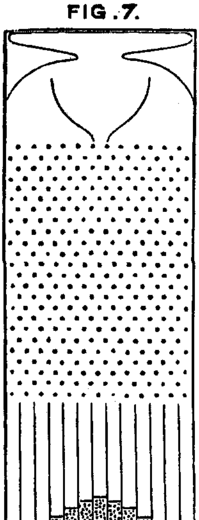


FIG. 8.

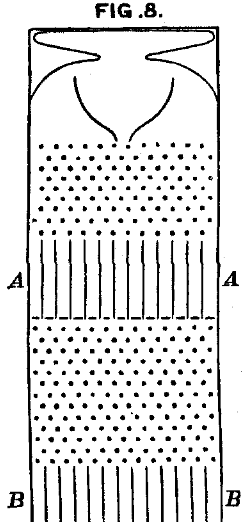
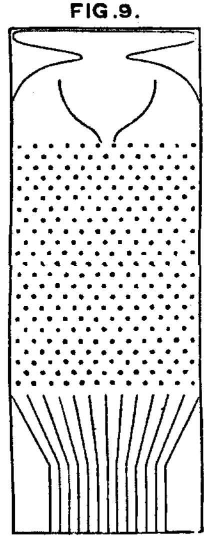


FIG. 9.



on which Deviation depends. It is a frame glazed in front, leaving a depth of about a quarter of an inch behind the glass. Strips are placed in the upper part to act as a funnel. Below the outlet of the funnel stand a succession of rows of pins stuck squarely into the back-board, and below these again are a series of vertical compartments. A charge of small shot is inclosed. When the frame is held topsy-turvy, all the shot runs to the upper end; then, when it is turned back into its working position, the desired action commences. Lateral strips, shown in the diagram, have the effect of directing all the shot that had collected at the upper end of the frame to run into the wide mouth of the funnel. The shot passes through the funnel and issuing from its narrow end, scampers deviously down through the pins in a curious and interesting way; each of them darting a step to the right or left, as the case may be, every time it strikes a pin. The pins are disposed in a quincunx fashion, so that every descending shot strikes against a pin in each successive row. The cascade issuing from the funnel broadens as it descends, and, at length, every shot finds itself caught in a compartment immediately after freeing itself from the last row of pins. The outline of the columns of shot that accumulate in the successive compartments approximates to the Curve of Frequency (Fig. 3, p. 38), and is closely of the same shape however often the experiment is repeated. The outline of the columns would become more nearly identical with the Normal Curve of Frequency, if the rows of pins were much more numerous, the shot smaller, and the compartments narrower; also if a larger quantity of shot was used.

The principle on which the action of the apparatus depends is, that a number of small and independent accidents befall each shot in its career. In rare cases, a long run of luck continues to favour the course of a particular shot towards either outside place, but in the large majority of instances the number of accidents that cause Deviation to the right, balance in a greater or less degree those that cause Deviation to the left. Therefore most of the shot finds its way into the compartments that are situated near to a perpendicular line drawn from the outlet of the funnel, and the Frequency with which shots stray to different distances to the right or left of that line diminishes in a much faster ratio than those distances increase. This illustrates and explains the reason why mediocrity is so common.

If a larger quantity of shot is put inside the apparatus, the resulting curve will be more humped, but one half of the shot will still fall within the same distance as before, reckoning to the right and left of the perpendicular line that passes through the mouth of the funnel. This distance, which does not vary with the quantity of the shot, is the "Prob: Error," or "Prob: Deviation," of any single shot, and has the same value as our Q. But a Scheme of Frequency is unsuitable for finding the values of either M or Q. To do so, we must divide its strangely shaped *area* into four equal parts by vertical lines, which is hardly to be effected except by a tedious process of "Trial and Error." On the other hand M and Q can be derived from Schemes of Distribution with no more trouble than is needed to divide a *line* into four equal parts.

Order in Apparent Chaos.—I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of *regularity* proves to have been latent all along. The tops of the marshalled row form a flowing curve of invariable proportions; and each element, as it is sorted into place, finds, as it were, a pre-ordained niche, accurately adapted to fit it. If the measurement at any two specified Grades in the row are known, those that will be found at every other Grade, except towards the extreme ends, can be predicted in the way already explained, and with much precision.

Problems in the Law of Error.—All the properties of the Law of Frequency of Error can be expressed in terms of Q, or of the Prob: Error, just as those of a circle can be expressed in terms of its radius. The visible Schemes are not, however, to be removed too soon from our imagination. It is always well to retain a clear geometric view of the facts when we are dealing with statistical problems, which abound with dangerous pitfalls, easily overlooked by the unwary, while they are cantering gaily along upon their arithmetic. The Laws of Error are beautiful in themselves and exceedingly fascinating to inquirers, owing to the thoroughness and simplicity with which they deal with masses of materials that appear at first sight to be entanglements on the largest scale, and of a hopelessly confused description. I will mention five of the laws.

(1) The following is a mechanical illustration of the first of them. In the apparatus already described, let q stand for the Prob: Error of any one of the shots that are dispersed among the compartments BB at its base. Now cut the apparatus in two parts, horizontally through the rows of pins. Separate the parts and interpose a row of vertical compartments AA, as in Fig. 8, p. 63, where the bottom compartments, BB, corresponding to those shown in Fig. 7, are reduced to half their depth, in order to bring the whole figure within the same sized outline as before. The compartments BB are still deep enough for their purpose. It is clear that the interpolation of the AA compartments can have no ultimate effect on the final dispersion of the shot into those at BB. Now close the bottoms of all the AA compartments; then the shot that falls from the funnel will be retained in them, and will be comparatively little dispersed. Let the Prob: Error of a shot in the AA compartments be called a. Next, open the bottom of any one of the AA compartments; then the shot it contains will cascade downwards and disperse themselves among the BB compartments on either side of the perpendicu-

lar line drawn from its starting point, and each shot will have a Prob: Error that we will call b. Do this for all the AA compartments in turn; b will be the same for all of them, and the final result must be to reproduce the identically same system in the BB compartments that was shown in Fig. 7, and in which each shot had a Prob: Error of q.

The dispersion of the shot at BB may therefore be looked upon as compounded of two superimposed and independent systems of dispersion. In the one, when acting singly, each shot has a Prob: Error of a; in the other, when acting singly, each shot has a Prob: Error of b, and the result of the two acting together is that each shot has a Prob: Error of q. What is the relation between a, b, and q? Calculation shows that $q^2 = a^2 + b^2$. In other words, q corresponds to the hypothenuse of a right-angled triangle of which the other two sides are a and b respectively.

(2) It is a corollary of the foregoing that a system Z, in which each element is the Sum of a couple of independent Errors, of which one has been taken at random from a Normal system A and the other from a Normal system B, will itself be Normal.[1] Calling the Q of the Z system q, and the Q of the A and B systems respectively, a and b, then $q^2 = a^2 + b^2$.

[1] We may see the rationale of this corollary if we invert part of the statement of the problem. Instead of saying that an A element deviates from its M, and that a B element also deviates independently from its M, we may phrase it thus: An A element deviates from its M, and its M deviates from the B element. Therefore the deviation of the B element from the A element is compounded of two independent deviations, as in Problem 1.

(3) Suppose that a row of compartments, whose upper openings are situated like those in Fig. 7, page 63, are made first to converge towards some given point below, but that before reaching it their sloping course is checked and they are thenceforward allowed to drop vertically as in Fig. 9. The effect of this will be to compress the heap of shot laterally; its outline will still be a Curve of Frequency, but its Prob: Error will be diminished.

The foregoing three properties of the Law of Error are well known to mathematicians and require no demonstration here, but two other properties that are not familiar will be of use also; proofs of them by Mr. J. Hamilton Dickson are given in Appendix B. They are as follows. I purposely select a different illustration to that used in the Appendix, for the sake of presenting the same general problem under more than one of its applications.

(4) Bullets are fired by a man who aims at the centre of a target, which we will call its M, and we will suppose the marks that the bullets make to be painted red, for the sake of distinction. The system of lateral deviations of these red marks from the centre M will be approximately Normal, whose Q we will call *c*. Then another man takes aim, not at the centre of the target, but at one or other of the red marks, selecting these at random. We will suppose his shots to be painted green. The lateral distance of any green shot from the red mark at which it was aimed will have a Prob: Error that we

will call b. Now, if the lateral distance of a particular green mark from M is given, what is the *most probable* distance from M of the red mark at which it was aimed? It is $\sqrt{\frac{c^2}{c^2+b^2}}$.

(5) What is the Prob: Error of this determination? In other words, if estimates have been made for a great many distances founded upon the formula in (4), they would be correct on the average, though erroneous in particular cases. The errors thus made would form a normal system whose Q it is desired to determine. Its value is $\frac{bc}{\sqrt{(b^2+c^2)}}$.

By the help of these five problems the statistics of heredity become perfectly manageable. It will be found that they enable us to deal with Fraternities, Populations, or other Groups, just as if they were units. The largeness of the number of individuals in any of our groups is so far from scaring us, that they are actually welcomed as making the calculations more sure and none the less simple.